Light Microscopy
Problem Set 2

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Please hand in the solutions of this problem set in the lecture on 12th of November 2018.

2 Lens as a Fourier Transformer(1)

Let's take a look again to the FOMO Script Chapter 4.1: Imaging of arbitrary optical field with thin lens.

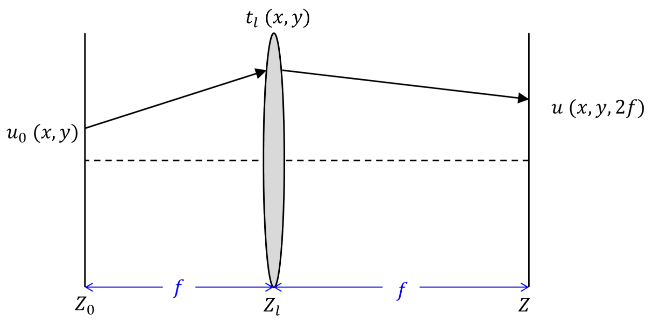

Figure 1: Geometry for optical propagation through a lens.

(a) Write down the transfer function or field transmission of a perfect thin lens. Explain all the variables you are using.

(b) Explain what the exponential term represents.

(c) Considering the field at plane Z_0 as $u(x, y)$, calculate the field distribution at the plane Z_l, $u_-(x, y, f)$. Hint: use the Fresnel diffraction integral.

(d) Calculate the field distribution after the interaction with the lens, $u_+(x, y, f)$.

(e) Now calculate the field distribution at Z.

3 Invariance requirements for similarity in imaging

The Abbe sine condition and the Herschel condition are invariance requirements that an optical system needs to fulfill, individually, to obtain an image similar to the object.

(a) Considering a constant transverse spatial frequency scaling, use the Fourier picture to derive a condition relating the angles of the k-vectors in object space to corresponding k-vectors in image space. Hence describe the obtained relation (Abbe sine condition) and its impact on the image.

(b) Derive a condition relating half of the previous mentioned angles to obtain a constant longitudinal spatial frequency magnification (Herschel condition).

(c) Under which condition can the Herschel and the Abbe sine condition both be fulfilled simultaneously? Which optical system fulfills these requirements?.

4 The Spherical Pupil and the Aplanatic Factor

Imagine half a spherical shell of (infinitesimal) thickness δr of constant density, being parallel-projected onto a plane.

(a) Calculate the projected (summed) density in dependence of the radius R from the center of the plane.

(b) Often, an objective is drawn as single line/plane at which the directions of light rays change. What are the problems with such a description for the purpose of imaging? How can this drawing be amended to be a more accurate model for light rays?

5 Bonus: Aberrations

In the lecture the transmission function or contrast transfer function of an ideal lens was defined by Equation (1).

Considering the pupil function P as complex, the real part describes absorption and the imaginary part describes the phase. Therefore, P can separated in two terms

$$P(\vec{\xi}) = A(\vec{\xi})e^{iW(\xi^2)} \tag{1}$$

where A and $W \in \mathbb{R}$. A is a damping function correspondent to incoherent aberrations.

The phase factor W is associated with coherent aberrations. One approach to describe Wavefront Aberrations is doing an expansion on an uniquely defined interval, the pupil function P, and choosing the **Zernike Polynomials** as a set which is orthogonal on this region.

Zernike Polynomials represent the expansion of the wave equation on a circular area. These expressions will be used for describing coherent aberrations on the image.

The expansion of the wave aberration on a circular area is

$$W(r,\phi) = \sum_n \sum_{m=-n}^{n} c_{nm} Z_n^m(r,\phi) \tag{2}$$

where the coefficients are

$$c_{nm} = \frac{2(n+1)}{\pi(1+\delta_{m0})} \int_0^1 \int_0^{2\pi} W(r,\phi) Z_n^m(r,\phi) r d\phi dr \tag{3}$$

and the definition of the Zernike Polynomials

$$Z_n^m(r,\phi) = \sqrt{\frac{2(n+1)}{1+\delta_{m0}}} R_n^m(r) \cdot \begin{Bmatrix} \sin(m\phi) & for \quad m<0 \\ \cos(m\phi) & for \quad m>0 \\ 1 \quad for \quad m=0 \end{Bmatrix} \tag{4}$$

here r and ϕ are the polar coordinates of the circular area (pupil). R_n^m are the radial polynomials

$$R_n^m(r) = \sum_{j=0}^{\frac{n-m}{2}} \frac{(-1)^j (n-j)!}{j!(\frac{n+m}{2}-j)!(\frac{n-m}{2}-j)!} r^{n-2j} \tag{5}$$

To picture the different wave aberrations, the ideal k-pupil $k_x^2 + k_y^2 = k_0^2$ was multiplied by a phasor term that includes Zernike polynomials $exp(iW_{nm})$.

The following Matlab code (DIPimage toolbox) represents this process.

```
n=256;
A=rr(n,n,'freq')<1/8 % real aperture
% Spherical aberration:
W= 6*(xx(n,n,'freq').^4+yy(n,n,'freq').^4) + ...
    12*xx(n,n,'freq').^2.*yy(n,n,'freq').^2 - ...
    6*(xx(n,n,'freq').^2+yy(n,n,'freq').^2) +1
c = 100; % Zernike coefficient
P = A * exp(i*c*W); % Complex valued pupil function
t = P .* exp(-i*pi*(2*rr(n,n,'freq')).^2) % Transmission function of a lens
PSF = ft(t) % Fourier Transform with a lens 2f
```

Modify the code above or use your own to illustrate how the following adjustment errors and phase modifications (wave aberrations) alter the ideal ideal pupil in Fourier space.

Report your code and the obtained PSF.

(a) Defocus (W_{20})

(b) Tip/Tilt (W_{11}, W_{1-1})

(c) Spherical Aberration (W_{40})

(d) Coma (W_{31}, W_{3-1})

(e) Astigmatism (W_{22}, W_{2-2})

(1) Script "Fundamentals of Modern Optics", FSU Jena, Prof. T. Pertsch.