3 Light in free space and at an interface

Most microscopy methods are based on what is called far-field optics. This means, that the optical elements (interfaces) are far away from the sample, at least if seen in the dimensions of the wavelength of light. This in turn means that we can describe the electromagnetic waves by only considering propagating waves, i.e. plane waves. This chapter thus tries to develop an intuitive understanding of such waves by considering their picture in Fourier-space. Snell's well known law of geometrical optics is easily obtained by a few considerations using Fouriers-space. The chapter finished by considering the electromagnetic field also in the vicinity of an optical inhomogeneity: The interface between two optical materials.

3.1 Free space

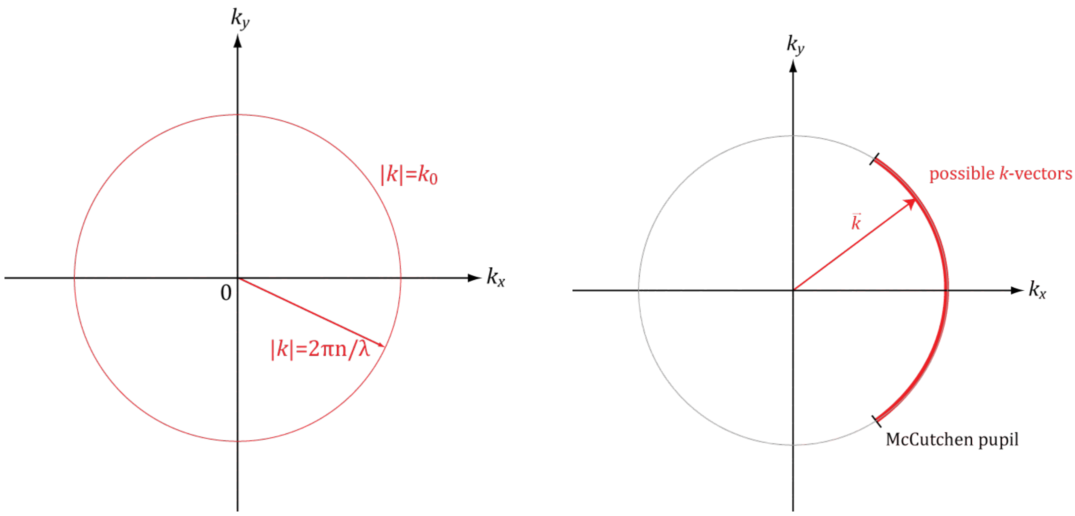

Figure 3.1: The k-sphere and the McCutchen pupil. A sphere of possible propagating monochromatic waves. The captured angles are limited by the opening angles of the microscope objective

- Free space is homogeneous, thus the Helmholtz equation applies.
- All propagating waves end up on the same sphere of $k_0 = |\mathbf{k}|$ vectors. We will use this sphere a lot. It is called the k-sphere (3.1), (slightly imprecise) the Ewald sphere, or the Descartes sphere. When the range of angles is limited as it is the

case for a typical microscope objective, the remaining spherical cap is called the *McCutchen pupil* [1].

- This principle applies also separately to each of the vector components. Thus we really have 3 such k_0-spheres that are not influencing another in homogeneous space
- Thus far-field optics is really simple. Just think about the sphere and you understand.
- A translation of our coordinate system in space means a multiplication with a phase ramp (see 2.7).

3.2 Free space propagation of light

The Fourier-Shift theorem (see 2.7) applied to the k_0-sphere. Shift by $\mathbf{\Delta Z}$ corresponds to multiplication with a pure phase-wave $\exp(i\mathbf{k}\mathbf{\Delta Z})$. Fig. 3.2 shows the effect that a shift of the coordinate system has on Fourier space. The k-sphere with equal phase zero is shown on the left, which would generate a perfect focus at position 0. If we now translate the coordinate system by ΔZ, this corresponds to a multiplication with a phase slope (middle). The resultant k-sphere now has an imprinted phase slope, which will generate the focus at the relative position $-\Delta Z$.

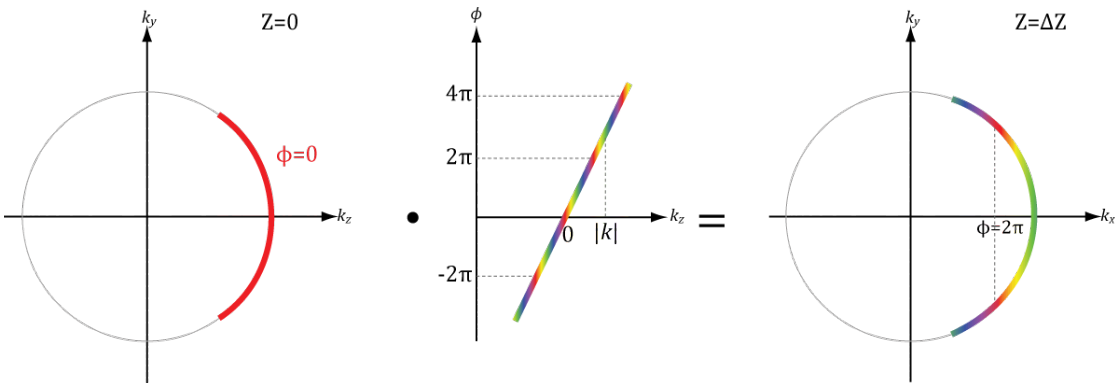

Figure 3.2: Real space shift imprinting a phase ramp in Fourier space.

- The Fourier projection theorem applied to 2D slicing in real space (or vice-versa). Note that the projection (sum over Z from 3D to 2D) automatically leads to a $\cos^2$ brightness increase for high angles on the 3D sphere or a brightness decrease for the projection from 2D to 3D.
- Free-space propagation of light by ΔZ along the z coordinate:
  1. Fourier-transforming a 2D real space slice.

2. Projecting the 2D Fourier-transform onto the 3D sphere of k_0-vectors.
3. Applying Z-translation by multiplying with the propagation phasor.
4. Projecting back to 2D.
5. Inverse-Fourier-transformation to get the new 2D slice in real space.

- Summarising steps 2, 3 and 4 by a 2D multiplication with a 2D propagation phasor:

$$Ph(k_x, k_y) = \exp(i\Delta Z k_z(k_x, k_y)) \tag{3.1}$$

$$= \exp(i\Delta Z\sqrt{{k_0}^2 - {k_x}^2 - {k_y}^2}) \tag{3.2}$$

where the theorem of Pythagoras was used: ${k_0}^2 = {k_x}^2 + {k_y}^2 + {k_z}^2$.
As a side-note: The spherical cap is curved as a sphere (see equation above). However, it is sometimes convenient to approximate this by a parabola:

$$k_z(k_x, k_y) = k_0 - \frac{{k_x}^2 + {k_y}^2}{|k_0|} \tag{3.3}$$

This is called the ***Fresnel approximation***. Analytically it is sometimes easier to perform analytical calculations using this expression, which is one reason why it is being used. Computationally, it turns out, that a single propagation which would require two Fourier-transforms can, with the help of this parabolic approximation been written as a single Fourier transform multiplying parabolic phasor terms to the 2D amplitude distribution before and after the transform: A complex valued field u(x,y) at Z-position 0 has to first be multiplied with a quadratic phase term

$$u'(x, y, 0) = \exp(i\frac{\pi}{\lambda\Delta Z})u(x, y, 0) \tag{3.4}$$

Then it is Fourier transformed:

$$u''(k_x, k_y) = \mathcal{F}\{u'(x, y, 0)\} \tag{3.5}$$

The the spatial frequencies are replaced by scaled spatial coordinates and the expression is multiplied with a final phase and amplitude factor accounting for curvature and dimming of the wave:

$$u''(x, y, \Delta Z) = \frac{\exp(ik\Delta Z)}{i\lambda\Delta Z}\exp(i\frac{\pi}{\lambda\Delta Z})u''(\frac{x}{\lambda\Delta Z}, \frac{y}{\lambda\Delta Z}) \tag{3.6}$$

Note that this procedure needs only one Fourier-transform instead of two as needed in the previously described way of propagating light from one plane to another. However, as soon as several planes need to be calculated (e.g. in holography) this advantage is mostly lost, since also in the previously described case only a single forward Fourier transformation would be needed from k-Space to each of the Z-positions to propagate to.

- Via the convolution theorem, propagation by ΔZ can also be interpreted as a convolution of the 2D amplitude distribution with a 2D (out-of-focus) amplitude spread function (ASF). The ASF can be obtained as the Fourier transformation of the above propagator.

- Summarizing the whole procedure by 2D convolving with a propagation function. This works in 2D and even in 3D for 3D emitter distributions.

- When computing this, there can be a domain and sampling problem. E.g. in imaging the waves are at high angle in the sample space (needing dense sampling of a small compact region) but low angle in the image space (needing sparse sampling of a large region). We will return to this effect later at the lens and in Gabor-inline holography.

3.3 Light in a medium

- The medium consists of a dense population of (Rayleigh) scatterers. Each scatterer emits a scattered wave at shifted phase with respect to the incoming wave. Energy preservation guarantees, that the resultant wave (superposition of incoming and outgoing waves) is of equal strength. Adding all emitted waves in a plane of equal phases yields an emitted (but phase delayed) plane wave of the same direction as the incoming wave. Overall, this results in a phase shift and then along the propagation direction in a change of the effective wavelength. This is summarized in the refractive index by $\lambda_{\text{medium}} = \lambda/n$.

3.4 Light at a planar interface

- The phase of waves is continuous across the surface. Therefore Snell's law is:

  $$\frac{\sin(\alpha_1)}{\sin(\alpha_2)} = \frac{n_2}{n_1} \tag{3.7}$$

  with the angles to the surface normal α_1 and α_2 and the corresponding refractive indices n_1 and n_2. This is also seen in Fourier space by the condition that the k-vector component along the surface orientation has to be identical on both sides with different radius of the Ewald sphere (Fig. 3.3).

- The amplitude (or the electric field) has to be continuous at the transition between the two materials. In Fourier space this means that the $\mathbf{k}$-vector components projected onto the surface have to be equal in both media (each with a different $\mathbf{k}$-sphere diameter $k_0 = |\mathbf{k}| = 2\pi n/\lambda$)

- Polarisation does matter!
  The waves leaving the surface have to be transversal. From the latter follow effects such as the Brewster angle (dielectrics) or the polarisation effects of high-angle

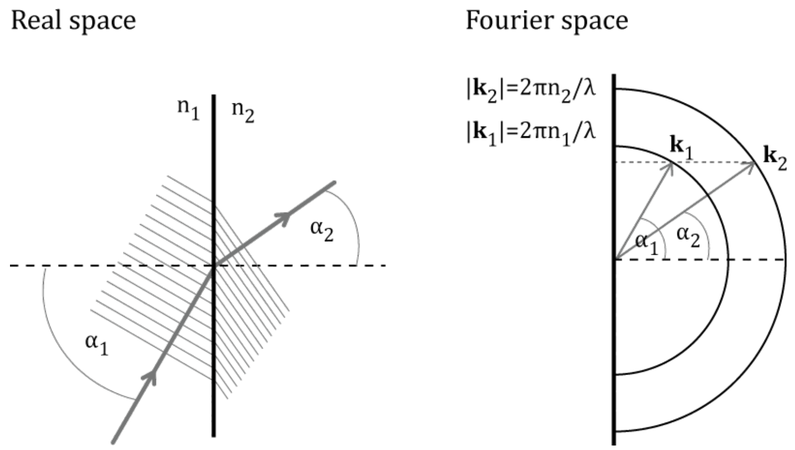

Figure 3.3: Snell's law

focusing (see below).
Fresnel's coefficients for transmission or reflection (e.g. $\sim 4\%$ reflection on glass for perpendicular incidence) can be derived form Maxwell's equations assuming no charges or currents at the interface. Multi-layer coating can allow close to 100% transmission.

3.5 Evanescent waves

Waves are of the form of Eqn. 1.10 which is the general solution of the Helmholtz-equation. However, we now consider the possibility for complex k-vector components in $\mathbf{k}$. $|\mathbf{k}|$ then still adheres to the condition $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$, but allowing some components to be complex has interesting implications. E.g. if we considers an imaginary valued k_z, $k_{radial} = \sqrt{k_x^2 + k_y^2}$ is then larger than the usual limit $k_0 = |\mathbf{k}|$. Thus in the projection (corresponding to a single slice in real space Fourier transformed) all k_x and k_y-vectors are now allowed, but for $k_{radial} > k_0$ we have imaginary values for k_z which are called **evanescent waves**, see Fig. 3.4 for an example of total internal reflection at a glass-air interface. These imaginary k-vectors turn this direction of a normally propagating plane wave into an exponentially decaying wave with no phase change along this direction, e.g. $e^{-|k_z|r_z - i(k_x r_x + k_y r_y)}$ for the case of total internal reflection. Note that the phase here is constant along the Z-direction r_z. In general it is also possible that multiple components of $\mathbf{k}$ are imaginary in value. Due to their exponential decay, evanescent waves can be neglected after a few μm distance from the sources. Plane waves are thus propagating waves and evanescent waves are not. The field ("near field") that surrounds an emitter in its vicinity has plane wave and evanescent components.

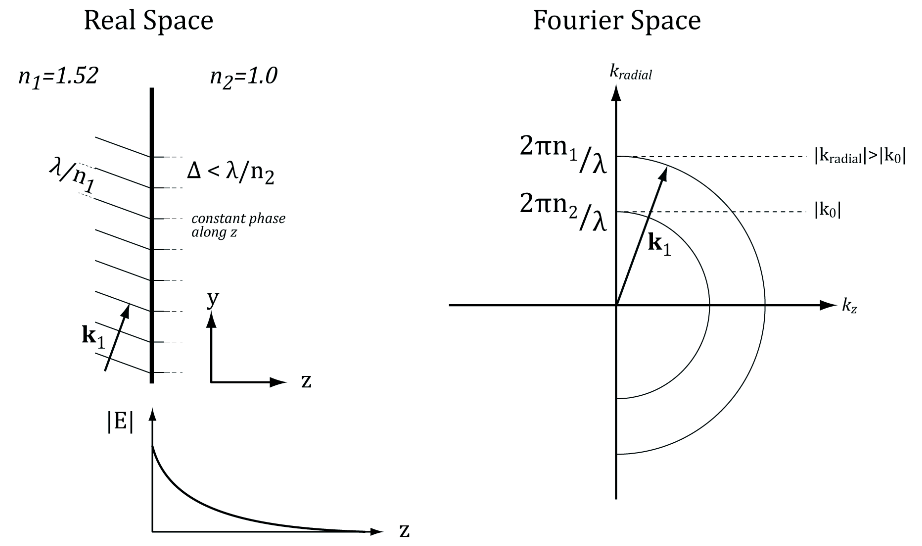

Figure 3.4: Evanescent waves by internal reflection. On the left side we see that it is, in contrast to Fig. 3.3, not possible to find a coarse enough wavelength in the optically less dense medium $n_2 = 1.0$ for a grazing incidence angle along the surface. The only possible solution is to introduce an imaginary k_z component such that $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$ still holds.

3.6 Literature and References

Literature

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References

[1] C. W. McCutchen. “Generalized aperture and the three-dimensional diffraction image: erratum”. In: *Journal of the Optical Society of America A* 19.8 (2002), p. 1721. ISSN: 1084-7529.