# 2 Short Introduction to Fourier Optics

A solution to electromagnetic waves in a homogeneous medium is a plane wave. For monochromatic light, such plane waves form a complete basis system and in linear optics any such field can simply be written as a sum of such basic plane waves. This is the reason that the Fourier-transformation, the operation that decomposes any function into a sum of such plane waves $e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ is very popular in optics. In this chapter, we thus review a number of interesting mathematical properties of the Fourier-transformation, many of which will be used in successive chapters in the context of optics, paving the way for a visual interpretations of seemingly complex properties of optical microscope methods.

The Fourier transformation is defined as

$$\mathcal{F}\{u(\mathbf{r})\}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(\mathbf{r})\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}\mathbf{r} \tag{2.1}$$

and its inverse transformation

$$\mathcal{F}^{-1}\{u(\mathbf{k})\}(\mathbf{r}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(\mathbf{k})\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}\mathbf{k} \tag{2.2}$$

Fourier space is also called reciprocal space, k-space or frequency space.

## 2.1 Properties of the Fourier-transformation

- Only really defined for square-integrable functions $u(\mathbf{r}) \in \mathcal{L}_2$
- The basis is simple in Fourier-space: a delta peak for propagating waves
- But with the help of distributions can be extended.
  e.g. constant $\rightarrow$ $\delta$ at zero (see [1])
- Linear
  $$\mathcal{F}\{au(\mathbf{r}) + bv(\mathbf{r})\} = a\mathcal{F}\{u(\mathbf{r})\} + b\mathcal{F}\{v(\mathbf{r})\}, \tag{2.3}$$
  with $a$ and $b$ being scalar values.
- Separable function remain separable
  $$\mathcal{F}\{u(x)v(y)\} = \mathcal{F}\{u(x)\}\mathcal{F}\{v(y)\} \tag{2.4}$$
- Space scaling property (illustrated in Fig. 2.1)
  $$\mathcal{F}\{u(\beta\mathbf{r})\}(\mathbf{k}) = \frac{1}{|\beta|}\mathcal{F}\{u(\mathbf{r})\}(\frac{1}{\beta}\mathbf{k}) \tag{2.5}$$

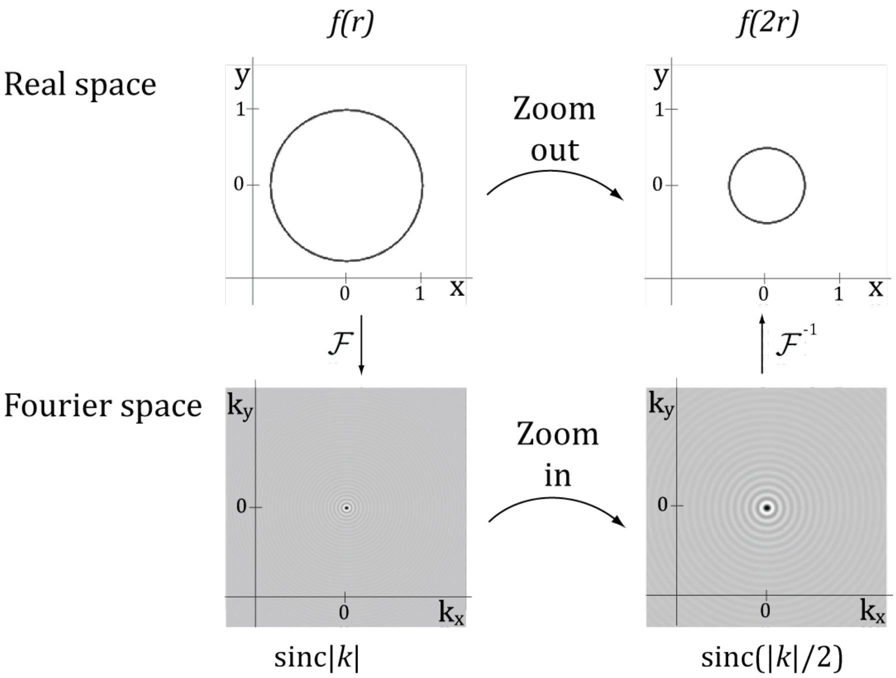


**Figure 2.1: Space scaling property**. An inverse relation exists between Fourier space and real space scaling. Here the circle function circ $= \delta(|\mathbf{r}| - 1)$ is used as an example.

- Symmetry properties
  - Real valued → Even function
  - Imaginary valued → Odd function

  Note that these properties apply both ways round for real space or Fourier-space denoting either of the sides.
- Transforming complex conjugates

$$\mathcal{F}\{u^*\}(\mathbf{k}) = (\mathcal{F}\{u\}(-\mathbf{k}))^* \tag{2.6}$$

- $\mathcal{F}\{\mathcal{F}\{f(\mathbf{r})\}\} = f(-\mathbf{r})$. After 4 transformations, you are back to $f$.
- Fourier shift theorem

$$\mathcal{F}\{u(\mathbf{r} + \Delta\mathbf{r})\} = \mathcal{F}\{u(\mathbf{r})\} \exp(-i\mathbf{k}\Delta\mathbf{r}) \tag{2.7}$$

- Spatial position information is mostly "coded" in the phase of the Fourier transformation (see Fourier shift theorem above).
- Zero frequency

$$\mathcal{F}\{f\}(\mathbf{k})|_{\mathbf{k}=\mathbf{0}} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{r})\, \mathrm{d}\mathbf{r} \tag{2.8}$$

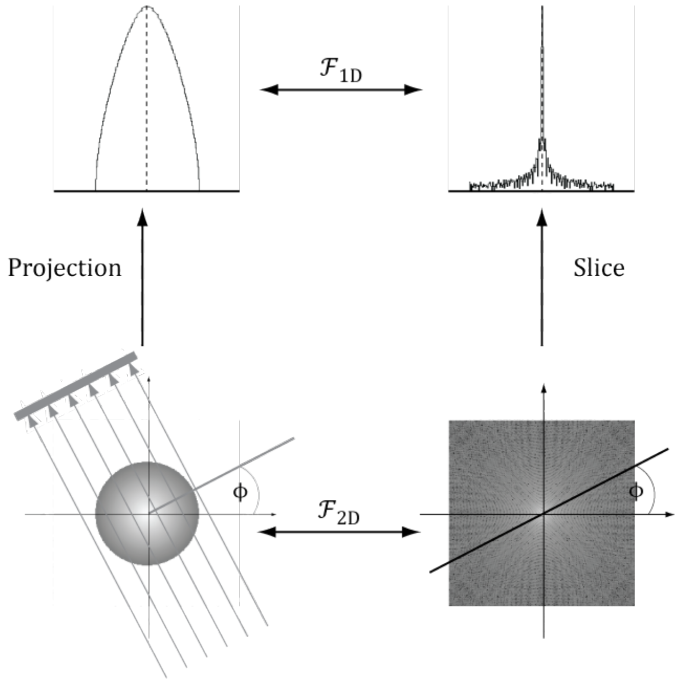


**Figure 2.2: The Fourier slice theorem (projection slice theorem)**

- Fourier slice theorem (projection slice theorem): Extracting a single slice in one space means summing over the direction orthogonal to the slice in reciprocal space.

$$\mathcal{F}_{xy}\{f(x,y,z=0)\}(\mathbf{k}_{xy}) = \int_{-\infty}^{\infty} \mathcal{F}\{f(x,y,z)\}(\mathbf{k})\,\mathrm{d}k_z \tag{2.9}$$

, where $\mathcal{F}_{xy}$ denotes the 2-dimensional Fourier transformation along x and y. More generally z can be any direction with a corresponding direction in Fourier space.

- *Parseval's theorem* (also called *Plancherel's theorem*)
$\rightarrow$ energy conservation

$$\int_{\mathbb{R}^n} |u(\mathbf{r})|^2\,\mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} |\mathcal{F}\{u\}(\mathbf{k})|^2\,\mathrm{d}\mathbf{k} \tag{2.10}$$

or more general:

$$\int_{\mathbb{R}^n} u(\mathbf{r})v^*(\mathbf{r})\,\mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} \mathcal{F}\{u\}(\mathbf{k})\,(\mathcal{F}\{v\}(\mathbf{k}))^*\,\mathrm{d}\mathbf{k} \tag{2.11}$$

- Differentiation

$$(\mathcal{F}\{(\frac{\partial}{\partial r_j})u\}) = (-ik_j)\mathcal{F}(u) \tag{2.12}$$

- Integration
$$\mathcal{F}\left\{\int_0^{r_j} u(\mathbf{r}')\mathrm{d}r_j' + C\right\} = \begin{cases} (i\frac{1}{|k_j|})\mathcal{F}(u), & k_j \neq 0 \\ C, & k_j = 0 \end{cases} \tag{2.13}$$

- Objects typically have their strongest contribution at zero frequency. Especially when the object is positive. The reason is that a self-similar object (Fig. 2.3) can be described as
$$\rho(x,y) = \beta\rho(\alpha(x+\Delta x), \alpha(y+\Delta y))). \tag{2.14}$$
Due to the Fourier-scaling law, we get the condition
$$|\mathcal{F}\{\rho\}(\mathbf{k})| = \frac{1}{\alpha}|\mathcal{F}\{\rho\}(\frac{1}{\alpha}\mathbf{k})|. \tag{2.15}$$
This condition can be fulfilled by a function of the shape
$$|\mathcal{F}\{\rho\}(\mathbf{k})| = \gamma\frac{1}{|k|^{\epsilon}}. \tag{2.16}$$
For natural objects possessing sharp edges and self-similarity, the decay is often approximated by $\epsilon = 2$. See also [2].

- Values add linearly, when they are always the same phase, but grow only with $\sqrt{N}$, if the phase is randomly oriented. This can be exploited to predict the Fourier-transformation of randomly located point objects, where only the zero frequency is proportional to the number of points, whereas all other frequencies have the same expectency value of $\sqrt{N}$ with random phase values.

- If a phase varies periodically, the contribution converges to zero in comparison to other non-varying terms.

- If the *support* of a function (non-zero region) in one space is finite, support of the Fourier transformation is infinite. There is no non-trivial monotonically decaying

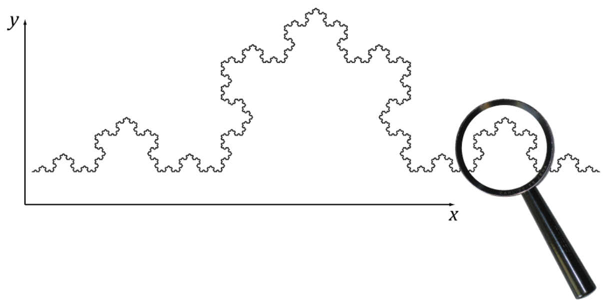


**Figure 2.3: Self-similar objects**. The magnified curve is a small copy of the whole curve: $\varrho(x,y) = \varrho(\alpha(x+\Delta x), \alpha(y+\Delta y))$.

function with finite support in its Fourier transformation. This is important when discussing image processing such as deconvolution, where the user would love to have a result that would be a positive monotonically decaying point spread functionh of the final filtered result. This is, however, impossible with a linear filter, since the support of the measured optical transfer function (OTF) is finite.

- Kramers–Kronig relationship relate the real and imaginary component of the electric susceptibility ($\chi(\omega) = \epsilon_r(\omega) - 1$) related to $n$ via $n = \sqrt{\epsilon_r \mu_r}$ to another:
  For
  $$\chi(\omega) = \chi_1(\omega) + i\chi_2(\omega)$$
  we get
  $$\chi_1(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega' - \omega}\, d\omega'$$
  and
  $$\chi_2(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_1(\omega')}{\omega' - \omega}\, d\omega',$$
  with the Cauchy principle value $\mathcal{P}$. This is due to causality in time [3]. This relationship offers insights into the interdependence of absorptive properties and refractive properties of a material, which are very intriguing. However, we mention this only for completeness and the Kramers–Kronig relationship is currently not needed for the understanding of the rest of this book.

## 2.2 Convolution

$$f(\mathbf{r}) \otimes g(\mathbf{r}) = \int_{\mathbb{R}^n} f(\mathbf{r}')g(\mathbf{r} - \mathbf{r}')\, \mathrm{d}\mathbf{r}' \tag{2.17}$$

The convolution is a "drawing with the brush" operation, where f is drawn with g as a brush or vice versa (same result). An alternative way of picturing convolution is to imagine the (2D) functions each to be printed out on a transparencies. One of them is then point mirrored at zero (flipped) and the transparencies are moved across each other and the total light transmitted through both of them is measured as a function of the shift.

- In Fourier space convolution is represented by a simple multiplication:
  $$\mathcal{F}\{f \otimes g\} = (2\pi)^{\frac{n}{2}}\, \mathcal{F}\{f\} \cdot \mathcal{F}\{g\} \tag{2.18}$$
  $$\text{and} \qquad \mathcal{F}\{f\} \otimes \mathcal{F}\{g\} = (2\pi)^{\frac{n}{2}} \mathcal{F}\{f \cdot g\} \tag{2.19}$$
  This is called the convolution theorem and will be used throughout this document.
- Commutativity: $f \otimes g = g \otimes f$

- Associativity: $f \otimes (g + h) = f \otimes g + f \otimes h$
- Associativity with complex number $\alpha$: $(\alpha f) \otimes g = \alpha(f \otimes g)$
- Convolution with a $\delta$ distribution: $f \otimes \delta = f$
- Complex conjuation $(f \otimes g)^* = (f^* \otimes g^*)$
- $\mathcal{F}\{f \otimes (g\, h)\} = \mathcal{F}\{f\}\,(\mathcal{F}\{g\} \otimes \mathcal{F}\{h\})$
- Convolution commutes with translation
- Cross–correlation

$$f(\mathbf{r}) \star g(\mathbf{r}) = \int_{\mathbb{R}^n} f^*(\mathbf{r}')g(\mathbf{r}' + \mathbf{r})\,\mathrm{d}\mathbf{r}' \tag{2.20}$$

  Here one can picture it similar to convolution, just without the need to flip one of the transparencies around. Cross-correlations are useful to find the position of an object (f) in the second image (g) by looking for places of local maxima (where maximal light gets transmitted through both transparencies, when the object to find in both of them overlaps.

- Auto–correlation

$$\mathcal{A}\{f\} = (f \star f)(\mathbf{r}') = \int_{-\infty}^{\infty} f^*(\mathbf{r})\; f(\mathbf{r} + \mathbf{r}')\,\mathrm{d}\mathbf{r}' = f^*(-\mathbf{r}') \otimes f(\mathbf{r}') \tag{2.21}$$

  also appears when squaring the absolute value in one space
  e.g. at the transition from amplitude to intensity:

$$|u(\mathbf{r})|^2 = u(\mathbf{r})u^*(\mathbf{r}) \tag{2.22}$$

  Since the complex conjugation of the last term above becomes in Fourier space a complex conjugated flipped version (see Eqn.2.6).Thus

$$\mathcal{F}\{|u(\mathbf{r})|^2\} = \mathcal{F}\{u\}(\mathbf{k}) \otimes \mathcal{F}^*\{u\}(-\mathbf{k}) = \mathcal{A}\{\mathcal{F}\{u\}\} \tag{2.23}$$

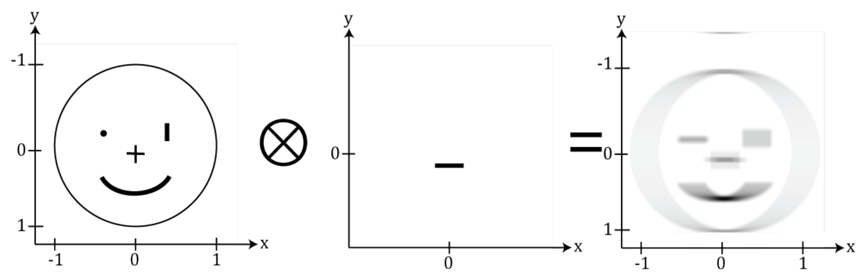


**Figure 2.4: Convolution**. Convolving two images can be understood as painting the first image with the second one as brush.

which is the auto-correlation of $\mathcal{F}\{u(\mathbf{r})\}$ or the "drawing with the brush" operation (convolution) of the Fourier transformation of $u$ with its flipped complex conjugate. Note also that

$$\mathcal{A}\{\rho(\mathbf{r}) \otimes h(\mathbf{r})\} = \mathcal{A}\{\rho(\mathbf{r})\}\mathcal{A}\{h(\mathbf{r})\} \tag{2.24}$$

| **Real space** | **Fourier space** |
|---|---|
| Plane wave | Delta peak |
| Delta peak | constant(if $\delta$ is at zero) or phase slope of constant amplitude |
| Delta line $\delta(x)\delta(y)$ | Delta plane $\delta(k_z)$ |
| Phase slope | Shifted delta peak<br>(leading to a shifted function, if phase slope is multiplied) |
| Gaussian | Gaussian (but remember the scaling law) |
| $\delta$-comb | $\delta$-comb (but remember the scaling law) |
| sin function | Two delta peaks |
| Rectangular box | sinc function |
| $\sin^2$ function | Three delta peaks |
| Triangular function | $\mathrm{sinc}^2$ function |
| $\mathrm{sgn}(r_j)$ | $\sqrt{\frac{2}{\pi}}\frac{1}{i\,k_j}$ |
| Decaying exponential<br>$\exp(-\beta x)(0 \text{ for } x < 0)$ | Lorentzian: $\frac{2\beta}{(2\pi k)^2+\beta^2}$ |
| 2D Box | 2D sinc |
| 2D Disc (radius $\alpha$) | jinc function (also called sombrero): $\frac{J_1(2\pi\alpha\lvert\mathbf{k}\rvert)}{2\pi\alpha\lvert\mathbf{k}\rvert}$ |
| 3D sperical shell<br>$(\delta\mathbf{r} - \mathbf{r_0})$ | scalar radial wave with no source: $\frac{\sin(2\pi\alpha\lvert\mathbf{k}\rvert)}{2\pi\alpha\lvert\mathbf{k}\rvert}$<br>spherical standing wave |
| 3D uniform sphere | $3\left(\frac{\sin(2\pi\alpha\lvert\mathbf{k}\rvert)}{(2\pi\alpha\lvert\mathbf{k}\rvert)^3} - \frac{\cos(2\pi\alpha\lvert\mathbf{k}\rvert)}{(2\pi\alpha\lvert\mathbf{k}\rvert)^2}\right)$ |
| 4D spherical shell | $2\,J_1(2\pi\alpha\lvert\mathbf{k}\rvert)/(2\pi\alpha\lvert\mathbf{k}\rvert)$ |
| 4D uniform sphere | $8\,J_2(2\pi\alpha\lvert\mathbf{k}\rvert)/(2\pi\alpha\lvert\mathbf{k}\rvert)$ |
| random arrangement of $\delta$-functions | random phases but absolute value is delta peak at zero frequency and an approximately constant for all other frequencies |
| White uncorrelated Gaussian noise | white uncorrelated Gaussian noise |
| Poisson noise of spatially varying strength | Noise of constant strength but the phase correlations contain the information where the noise originated |

**Table 2.1:** Table of Fourier-transformation pairs or correspondences (always applicable)

| **Real space** | **Fourier space (or vice versa)** |
|---|---|
| Multiplication with a phase slope | Shift of coordinate system |
| Multiplication with $\delta$-plane at $Z = 0$ (extracting a section) | 2D Projection orthogonal to line |
| Convolution with a PSF | Multiplication with its Fourier-transformation, the OTF |
| Rotation (by Euler angles) | Rotation (by the same angles) |
| Stretching by $\alpha$ along an axis | shrinking (stretching by $\frac{1}{\alpha}$ along the same axis) |
| Multiplying by $\delta$-comb (i.e. sampling) | Convolving with the N-dimensional $\delta$-comb (i.e. can lead to aliasing effects) $\rightarrow$ Nyquist limit |
| Derivative along $\mathbf{r}_j$: $\frac{\partial}{\partial \mathbf{r}}$ | Multiplication with $i\mathbf{k}_j$ |

**Table 2.2:** Operations in Real-space and Fourier-space

## 2.3 Literature and References

### Literature

- „*Introduction to Fourier Optics*“ – Goodman (2005), Roberts and Company Publishers
- „*Fourier Optics - An Introduction*“ Steward (2004), Dover Publications
- „*Optics*“ Hecht (2002), Addison Wesley

### References

[1] F. Laloe C. Cohen-Tannoudji B. Dio. *Quantum Mechanics.* Wiley, 1977.

[2] W. H. Hsiao and R. P. Millane. “Effects of occlusion, edges, and scaling on the power spectra of natural images”. In: *J. Opt. Soc. Am. A* 22.9 (2005), pp. 1789–1797.

[3] M. Schönleber. *A simple derivation of the Kramers-Kronig relations from the perspective of system theory.* checked: 02. Nov 2016. URL: http://www.iam.kit.edu/wet/plainhtml/Download/Derivation_Kramers-Kronig.pdf.