1 Introduction - Understanding the Nature of a light-wave

In this chapter the mathematical and theoretical physics groundwork will be presented. From the mathematical side, there is a special focus on complex numbers and Fourier-transformations, since these are tools which make the description of light and its properties amazingly simple to describe. Especially a good knowledge of the various Fourier-theorems can be an invaluable tool to ease the detailed theoretical understanding of various microscopy methods and modes. From the point of theoretical physics a short introduction is given, which sets the framework about the approximations which are done starting from Maxwell's equations to justify the description of various far-field microscopy systems in later chapters.

1.1 Complex Numbers

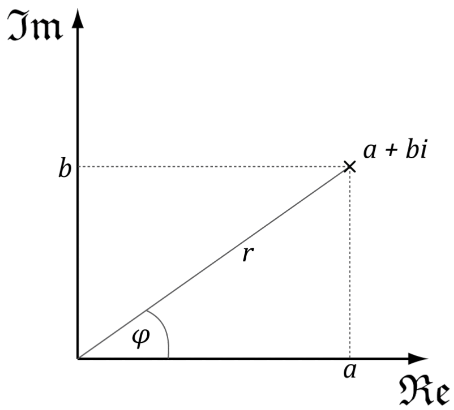

Figure 1.1: Argand diagram in Cartesian & polar coordinates. A complex number can be visually represented as a pair of numbers forming a vector on a diagram called an Argand diagram, representing the complex plane.

- **real and imaginary part**

$$z = a + b \cdot i = \Re(z) + \Im(z) \cdot i,$$

where a and b are real numbers and i is the imaginary unit, which satisfies the equation $i^2 = -1$

- **complex conjugate**

$$z^* = a - b \cdot i = \mathfrak{Re}(z) - \mathfrak{Im}(z) \cdot i$$

Geometrically, z^* is the "reflection" of z about the real axis. Conjugating twice gives the original complex number: $(z^*)^* = z$.

- **addition** corresponds to vector addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

- **polar and exponential form**
r and φ give another way of representing complex numbers, the polar form, as the combination of modulus and argument fully specify the position of a point on the plane. The absolute value (magnitude) of a complex number $z = x + yi$ is

$$r = |z| = \sqrt{x^2 + y^2}.$$

Recovering the original rectangular co-ordinates from the polar form is done by the trigonometric form

$$z = r(\cos\varphi + i\sin\varphi),$$

using Euler's formula $e^{ix} = \cos x + i \sin x$ this can be written as

$$z = re^{i\varphi} = |z|e^{i\,\mathrm{atan}_2(\mathcal{I}(z),\mathcal{R}(z))}.$$

where atan_2 refers to the phase of the complex valued quantity z as calculated by the *atan*2 function as defined in the C programming reference.

- **multiplication**:

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$

Polar coordinates: angles add, absolute values multiply.

- **division**:

$$\frac{a + bi}{c + di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i$$

Polar coordinates: angles subtract, absolute values divide.

- **complex roots**: Easiest understood in the polar notation: Divide the angle by the root exponent β and take the βth root of the absolute value. With an integer β, there will be β solutions equally distributed along the circle of constant amplitude.

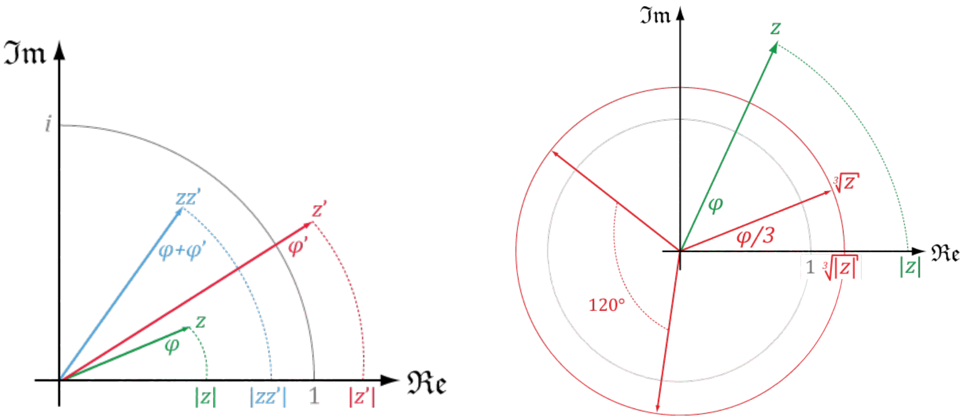

Figure 1.2: Complex multiplication & roots. Multiplication: Angles add, absolute values multiply. Roots: The n^{th} root has n solutions

1.2 Homogeneous Maxwell equations

Maxwell's equation were a fantastic summary of the physics known at the time. They managed to connect various known effects into a common framework. In the context of relativity theory more compact forms were devised, yet they are still mostly known as presented below. Since most phenomena regarding light are governed by them, they form the start of a short theoretical excursion. By making simplifying assumptions such as a homogeneous medium, linear optics and a single frequency simplified equations can be derived as shown below. However, it is also important to describe what is not governed by this system of equations: Maxwell's equations do not tell us about the existence of photons and their (strange) quantum dynamical properties such as entanglement.

$$\begin{aligned}
\text{Gauss' Law for electricity:} \quad & \operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0} \\
& \textit{Sources of the field.} \\
& \textit{Not present inside homogeneous media.} \\
\text{Gauss' Law for magnetism:} \quad & \operatorname{div} \mathbf{B} = 0 \\
& \textit{There are no magnetic monopoles.} \\
\text{Faraday's Law:} \quad & \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\text{Maxwell-Ampère's Law:} \quad & \operatorname{rot} \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\
& \textit{Currents as well as changing electric field} \\
& \textit{both induce a magnetic field around them.}
\end{aligned}$$

Convenient notation with the Nabla operator:

$$\text{Gauss' Law for electricity: } \vec{\nabla} \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0} \tag{1.1}$$

$$\text{Gauss' Law for magnetism: } \vec{\nabla} \cdot \mathbf{B} = 0 \tag{1.2}$$

$$\text{Faraday's Law: } \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.3}$$

$$\text{Maxwell-Ampère's Law: } \vec{\nabla} \times \mathbf{B} = \mu_0 \, j + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.4}$$

Note that these equations are all in vacuum. In a homogeneous medium μ_0 and ε_0 have to be replaced with material constants. The Maxwell equations cover effects such as magnetic effects and the near field. They can also deal with nonlinear material properties. They can be simplified by are assuming a homogeneous medium,an isotropic medium, and propagating (far-) fields only. Such simplifications as given below can often be used with great effect to describe the essence of a microscopic imaging method.
There are also integral forms of the Maxwell equations (see [1]) using Gauss's integral theorem and Stoke's theorem, but we are not using these here.

1.3 Wave equation and Helmholtz equation

The **wave equation** can be obtained from Maxwell's equations for $\mathbf{j} = 0$. Apply **rot** on both sides:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = -\vec{\nabla} \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \mathbf{B}\right) \tag{1.5}$$

$$= \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1.6}$$

$$\vec{\nabla} \times \vec{\nabla} \times \mathbf{E} = \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{E}\right) - \vec{\nabla} \cdot \left(\vec{\nabla}\mathbf{E}\right) \tag{1.7}$$

$$= \mathbf{grad}\,(\operatorname{div} \mathbf{E}) - \operatorname{div}\,(\mathbf{grad}\,\mathbf{E}) \tag{1.8}$$

with $\varrho = 0 \Rightarrow \operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_0} = 0$ and $\operatorname{div}\,(\mathbf{grad}\,\mathbf{E}) = \nabla^2 \mathbf{E}$:

$$\left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r}, t) = 0 \tag{1.9}$$

A solution is the plane wave:

$$\mathbf{E}(\mathbf{r}, t) = U_0 \, \exp(i(\mathbf{kr} - \omega t)) = \exp(i\mathbf{kr}) \exp(-i\omega t) \tag{1.10}$$

What we really mean is $E_x = \mathfrak{Re}(\exp(i(\mathbf{kr} - \omega t))$. This is essentially a mathematical trick. Therefore you should be aware, that this trick only works as long as you do not start to square things or alike. In many cases we are only interested in a single frequency (a monochromatic wave). Thus we can introduce the time independent quantity $\mathbf{u}$ related to the time dependent electric field via $\mathbf{E} = \mathbf{u}\, \exp(-i\omega t)$, which yields the homogeneous **Helmholtz equation**:

$$\vec{\nabla}^2 \mathbf{u} + k_0 \mathbf{u} = 0 \tag{1.11}$$

However, in many cases it is enough to consider only one of its components as a scalar value. For low angles this can also be interpreted as the electric field component such as $\mathbf{E}_x$ of linearly polarized light. However, these equations also apply separately for each electric field component.
Two particurly interesting solutions of the homogeneous Helmholtz equations are the plane wave:

$$\mathbf{u}(\mathbf{r}) = \mathbf{u_0}\, \exp(i\mathbf{kr}) \tag{1.12}$$

and the spherical wave:

$$\mathbf{u}(\mathbf{r}) = \mathbf{u_0}\, \frac{\exp(i|\mathbf{k}||\mathbf{r}|)}{|\mathbf{r}|} \tag{1.13}$$

in any region of space not containing the origin $|\mathbf{r}| = 0$.Note that the complex valued amplitude $\mathbf{u_0}$ can contain phase shifts between vector components, thus being capable of representing arbitrarycoherent states of polarisation.

Limitations

- No sources (emitting light) and no drains (absorbing light)
- The equation is a simplification for homogeneous isotropic media. Examples are free space (i.e. the space between the lenses) or space filled with water or immersion oil. Sometime it can also be assumed that the sample itself is "free space": if the index of refraction remains constant and there is no absorption.
- If we want the wave to oscillate with frequency ω, then $|\mathbf{k}| = \frac{\omega}{c}$ (dispersion relation of plane waves) or $\mathbf{k}$ is a complex valued vector (evanescent waves)

Further simplifications and consequences

- The scalar limit (apply Helmholtz to a scalar instead of a vector field)
- The one frequency limit: Only one ω is allowed. It follows everything is coherent
- Linear optics: The ωt-term can be avoided, remembering that it has to be written behind. Often a global phase is also neglected.
- Incoherence and partial coherence as sum of intensities but each with the same ω.
- Approximation: Integration time $T_{int} \gg \frac{1}{\Delta\omega}$
- $I \sim |E|^2 = EE^*$

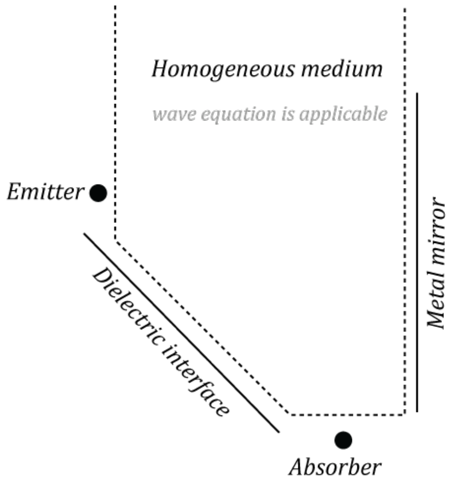

Figure 1.3: Homogeneous medium. This figure exemplifies what is meant with a region of space to which the wave equation (or the Helmholtz-equation) can be applied. This region has to be free of sources, free of sinks (here the absorber) or dielectric or metallic interfaces.

1.4 Superposition principle

- Any solutions of the Helmholtz equation can be linearly superimposed with any other solution.
- This holds for spatial as well as for temporal aspects of the solution. E.g. Waves of different k-vectors can be superimposed just as well as waves of different temporal frequencies.
- Plane waves (eqn. 1.12) and evanescent waves (eqn. 1.12, with complex valued components of $\mathbf{k}$) form a complete basis system.

1.5 Literature and References

Literature

- „*Fundamentals of Photonics*“Saleh, Teich (2007), Wiley
- „*Classical Electrodynamics*“Jackson, (1999), Wiley
- „*Fourier Optics and Computational Imaging*“Khare, (2015), Wiley
- „*Handbook of Optical Systems*, Vol.2, Physical Image Formation“ Gross, Totzeck, Singer (2005), Wiley-VCH
- „*The strange Theory of Light and Matter*“Feynmann (2006), Princeton University Press

- „*Principles of Optics*"Born, Wolf (1980), Cambridge University Press

References

[1] M. Sands R. P. Feynman R. B. Leighton. *The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter*. Basic Books, 2011.