Light Microscopy
Problem Set 3

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dedicated to ℝené ℝichter.

November 28, 2017

Please hand in the solutions of this problem set in the lecture on 28th of November 2017.

8 Lens as a Fourier Transformer (1)

The field transmission of a perfect thin lens is

$$t(\vec{\xi}) = P(\vec{\xi})e^{-i\pi\frac{k}{f}\xi^2} \tag{1}$$

where ξ are the lens plane lateral coordinates x, y and P is the pupil function or mask.

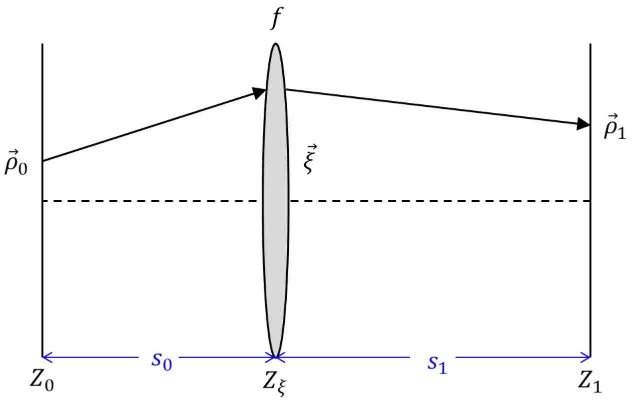

Figure 1: Geometry for optical propagation through a lens.

(a) Explain what the exponential term represents.

(b) Considering the field at plane Z_0 as $E(\vec{\rho_0}, z_0) = E_0(\vec{\rho_0})$, calculate the field distribution at the plane ξ, $E_\xi(\vec{\xi})$. Hint: use the Fresnel diffraction integral (2).

$$E(\vec{\rho}, z) = -i\frac{k}{z}e^{i2\pi kz}\int E(\vec{\rho_0}, 0)e^{i\pi\frac{k}{z}|\vec{\rho}-\vec{\rho_0}|^2}d^2\vec{\rho_0}, \tag{2}$$

(c) Again with with the help of the Fresnel diffraction integral calculate the field distribution in an arbitrary plane Z_1.

(d) Now calculate the field distribution at Z_1 for $s_1 = f$. Hint: apply the following integral

$$\int e^{i\alpha\rho_a^2 + i\beta\vec{\rho_a}\cdot\vec{\rho_b}}d^2\vec{\rho_a} = \frac{i\pi}{\alpha}e^{-\frac{i\beta^2\rho_b^2}{4\alpha}} \tag{3}$$

This assumes $Im[\alpha] > 0$ or ρ_a bounded.

9 Aberrations

In the lecture the transmission function or contrast transfer function of an ideal lens was defined by Equation (1).

Considering the pupil function P as complex, the real part describes absorption and the imaginary part describes the phase. Therefore, P can separated in two terms

$$P(\vec{\xi}) = A(\vec{\xi})e^{iW(\xi^2)} \tag{4}$$

where A and $W \in \mathbb{R}$. A is a damping function correspondent to incoherent aberrations.

The phase factor W is associated with coherent aberrations. One approach to describe Wavefront Aberrations is doing an expansion on an uniquely defined interval, the pupil function P, and choosing the **Zernike Polynomials** as a set which is orthogonal on this region.

Zernike Polynomials represent the expansion of the wave equation on a circular area. These expressions will be used for describing coherent aberrations on the image.

The expansion of the wave aberration on a circular area is

$$W(r,\phi) = \sum_{n} \sum_{m=-n}^{n} c_{nm} Z_n^m(r,\phi) \tag{5}$$

where the coefficients are

$$c_{nm} = \frac{2(n+1)}{\pi(1+\delta_{m0})} \int_0^1 \int_0^{2\pi} W(r,\phi) Z_n^m(r,\phi) r d\phi dr \tag{6}$$

and the definition of the Zernike Polynomials

$$Z_n^m(r,\phi) = \sqrt{\frac{2(n+1)}{1+\delta_{m0}}} R_n^m(r) \cdot \begin{Bmatrix} \sin(m\phi) & for & m<0 \\ \cos(m\phi) & for & m>0 \\ 1 & for & m=0 \end{Bmatrix} \tag{7}$$

here r and ϕ are the polar coordinates of the circular area (pupil). R_n^m are the radial polynomials

$$R_n^m(r) = \sum_{j=0}^{\frac{n-m}{2}} \frac{(-1)^j (n-j)!}{j!(\frac{n+m}{2}-j)!(\frac{n-m}{2}-j)!} r^{n-2j} \tag{8}$$

To picture the different wave aberrations, the ideal k-pupil $k_x^2 + k_y^2 = k_0^2$ was multiplied by a phasor term that includes Zernike polynomials $exp(iW_{nm})$.

The following Matlab code (DIPimage toolbox) represents this process.

```
n=256;
A=rr(n,n,'freq')<1/8 % real aperture
% Spherical aberration:
W= 6*(xx(n,n,'freq').^4+yy(n,n,'freq').^4) + ...
    12*xx(n,n,'freq').^2.*yy(n,n,'freq').^2 - ...
    6*(xx(n,n,'freq').^2+yy(n,n,'freq').^2) +1
c = 100; % Zernike coefficient
P = A * exp(i*c*W); % Complex valued pupil function
t = P .* exp(-i*pi*(2*rr(n,n,'freq')).^2) % Transmission function of a lens
PSF = ft(t) % Fourier Transform with a lens 2f
```

Modify the code above or use your own to illustrate how the following adjustment errors and phase modifications (wave aberrations) alter the ideal ideal pupil in Fourier space.

Report your code and the obtained PSF.

(a) Defocus (W_{20})

(b) Tip/Tilt (W_{11}, W_{1-1})

(c) Spherical Aberration (W_{40})

(d) Coma (W_{31}, W_{3-1})

(e) Astigmatism (W_{22}, W_{2-2})

(1) "Introduction to Optical Microscopy" by J. Mertz (Roberts and Company, 2010).